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Best Error Bounds for Derivatives in Two Point Birkhoff Interpolation Problems

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INTRODUCTION

Let $u \in C^{2m}[0, h]$ be given, let v_{2m-1} be the unique Hermite interpolation polynomial of degree $2m-1$ matching u and its first $m-1$ derivatives $u^{(j)}$ at 0 and h and let $e = v_{2m-1} - u$ be the error. Ciarlet *et al.* [3, Theorem 9] have obtained pointwise bounds on the error $e(x)$ and its derivatives in terms of $U = \max_{0 \leqslant x \leqslant h} |u^{(2m)}(x)|$. Their bounds are

$$|e^{(k)}(x)| \leqslant \frac{h^k (x(h-x))^{m-k}\, U}{k!\,(2m-2k)!}, \qquad k = 0, 1, ..., m; \quad 0 \leqslant x \leqslant h. \tag{1.1}$$

These bounds are best possible for $k = 0$ only. Later, in 1967, Birkhoff and Priver [1] obtained, for $m = 2$ and $m = 3$, optimal error bounds on the derivatives $e^{(k)}(x)$. More precisely, their results can be described by the following

THEOREM A. *Let* $u(x) \in C^4[0, h]$. Then

$$|v_3^{(k)}(x) - u^{(k)}(x)| \leqslant \alpha_k h^{4-k} \max_{0 \leqslant x \leqslant h} |u^{(4)}(x)|, \qquad k = 0, 1, 2, 3, \tag{1.2}$$

where

$$\alpha_0 = \frac{1}{4^2 4!}, \qquad \alpha_1 = \frac{\sqrt{3}}{216}, \qquad \alpha_2 = \frac{1}{12}, \qquad \alpha_3 = \frac{1}{2}. \tag{1.3}$$

Further, for $u(x) \in C^6[0, h]$, *we have*

$$|v_5^{(k)}(x) - u^{(k)}(x)| \leqslant \beta_k h^{6-k} \max_{0 \leqslant x \leqslant h} |u^{(6)}(x)|, \tag{1.4}$$

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where

$$\beta_0 = \frac{1}{4^3 6!}, \qquad \beta_1 = \frac{\sqrt{5}}{30000}, \qquad \beta_2 = \frac{1}{1920},$$
$$\beta_3 = \frac{1}{120}, \qquad \beta_4 = \frac{1}{10}, \qquad \beta_5 = \frac{1}{2}. \tag{1.5}$$

Here $v_3(x)$ and $v_5(x)$ are Hermite interpolation polynomials of degree $\leqslant 3$ and of degree $\leqslant 5$, respectively.

They also noted that, for $m > 3$, their method, using Green's function, seems unlikely to be useful. Analogously to using Hermite interpolation polynomials, one may choose to approximate a given function $u(x) \in C^{2m}[0, h]$ by the so called Lidstone interpolation polynomial (see [4, p. 28]) $L_{2m-1}(x)$ of degree $\leqslant 2m-1$, matching u and its first $m-1$ even derivatives $u^{(2j)}$ at 0 and h. It turns out that in this case we can give pointwise bounds on the error and its derivatives in terms of $u = \max_{0 \leqslant x \leqslant h} |u^{(2m)}(x)|$ which are also optimal. An important role in our Theorem 1 (see below) is played by the polynomial $Q_{2m}(x)$ (Euler polynomial) of degree $2m$ given by the formula

$$Q_{2m}(x) = -\int_0^1 G_1(x, t)\, Q_{2m-2}(t)\, dt, \qquad m = 1, 2, ..., \tag{1.6}$$

where

$$Q_0(x) = 1 \tag{1.7}$$

and

$$\begin{aligned} G_1(x, t) &= t(x-1), \qquad 0 \leqslant t < x \leqslant 1, \\ &= x(t-1), \qquad 0 \leqslant x \leqslant t \leqslant 1. \end{aligned} \tag{1.8}$$

Clearly from (1.6)–(1.8) it follows that

$$Q''_{2n}(x) = -Q_{2n-2}(x), \qquad Q_{2n}(0) = Q_{2n}(1) = 0.$$

Also

$$\begin{aligned} &Q_{2n}^{(2p)}(0) = Q_{2n}^{(2p)}(1) = 0, \qquad p = 0, 1, ..., n-1, \\ &Q_{2n}^{(2n)}(1) = Q_{2n}^{(2n)}(0) = (-1)^n, \\ &Q_{2n}^{(2j)}(x) = (-1)^j\, Q_{2n-2j}(x). \end{aligned} \tag{1.9}$$

Explicit representation of some of these polynomials is given by

$$Q_2(x)=\frac{x(1-x)}{2!},\qquad Q_4(x)=\frac{x^2(1-x)^2+x(1-x)}{4!},$$

$$Q_6(x)=\frac{x^3(1-x)^3+3x^2(1-x)^2+3x(1-x)}{6!},$$

$$Q_8(x)=\frac{x^4(1-x)^4+6x^3(1-x)^3+17x^2(1-x)^2+17x(1-x)}{8!}.$$

We now state our first result as follows.

THEOREM 1. *Let* $u(x)\in C^{2m}[0,1]$, *let* $L_{2m-1}(u,x)=L_{2m-1}(x)$ *be the unique polynomial of degree* $\leqslant 2m-1$ *satisfying the conditions*

$$L_{2m-1}^{(2j)}(0)=u^{(2j)}(0),\qquad L_{2m-1}^{(2j)}(1)=u^{(2j)}(1),\qquad j=0,1,\dots,m-1. \tag{1.10}$$

Then, for $0\leqslant x\leqslant 1$, *with* $u=\max_{0\leqslant x\leqslant 1}|u^{(2m)}(x)|$,

$$|u^{(2j)}(x)-L_{2m-1}^{(2j)}(x)|\leqslant uQ_{2m-2j}(x),\qquad j=0,1,\dots,m-1, \tag{1.11}$$

and

$$\begin{aligned}|u^{(2j-1)}(x)-L_{2m-1}^{(2j-1)}(x)|&\\ \leqslant u[(1-2x)\,Q'_{2m+2-2j}(x)+2Q_{2m+2-2j}]&\\ \leqslant Q'_{2m+2-2j}(0),\qquad j=1,2,\dots,m.&\end{aligned} \tag{1.12}$$

Moreover (1.11) *and* (1.12) *are best possible.*

Note! For $j=0$, (1.11) is implicitly contained in Theorem 1.1 and Theorem 2.1 of Widder [5].

Our next aim is to give some applications of Theorem A of Birkhoff and Priver to two point Birkhoff interpolation problems. For this purpose, let $f\in C^6[0,1]$ and let $H_5[f,x]$ be the unique polynomial of degree $\leqslant 5$ satisfying the conditions

$$H_5^{(p)}(f,x_i)=f^{(p)}(x_i),\qquad i=0,1,\;\; p=0,2,3;\qquad x_0=0,\;\; x_1=1. \tag{1.13}$$

we may call it the (0, 2, 3) interpolation polynomial with nodes 0 and 1. Concerning $H_5(f,x)$ we now state the following theorem.

THEOREM 2. *Let f,* $C^6[0, 1]$ *and let* $H_5[f, x]$ *satisfy* (1.13). *Then for* $0 \leqslant x \leqslant 1$,

$$|H_5^{(p)}(f, x) - f^{(p)}(x)| \leqslant u \max_{0 \leqslant x \leqslant 1} |f_0^{(p)}(x)|, \qquad p = 0, 1, ..., 5, \tag{1.14}$$

where

$$u = \max_{0 \leqslant x \leqslant 1} |f^{(6)}(x)|, \qquad f_0(x) = \frac{x^3(1-x)^3 + \frac{1}{2}x^2(1-x)^2 + \frac{1}{2}x(1-x)}{6!}. \tag{1.15}$$

Note. If we denote $c_p = \max_{0 \leqslant x \leqslant 1} |f_0^{(p)}(x)|$ then

$$c_0 = \frac{11}{64} \frac{1}{6!}, \qquad c_1 = \frac{1}{2} \frac{1}{6!}, \qquad c_{p+2} = \alpha_p, \qquad p = 0, 1, 2, 3,$$

where α_p are defined by (1.3).

Similarly let $f \in C^8[0, 1]$. We denote by $H_7[f, x]$ the unique polynomial of degree $\leqslant 7$ satisfying the conditions

$$H_7^{(p)}(f, x_i) = f^{(p)}(x_i), \qquad p = 0, 2, 3, 4, \quad i = 0, 1 \tag{1.16}$$

with $x_0 = 0$, $x_1 = 1$.

Concerning $H_7[f, x]$ we shall prove the following:

THEOREM 3. *Let* $f \in C^8[0, 1]$ *and* $H_7[f, x]$ *be the unique polynomial of degree* $\leqslant 7$ *satisfying* (1.16). *Then*

$$|H_7^{(p)}[f, x] - f^{(p)}(x)| \leqslant u_1 \max_{0 \leqslant x \leqslant 1} |f_1^{(p)}(x)|, \qquad p = 0, 1, ..., 7, \tag{1.17}$$

$$u_1 = \max_{0 \leqslant x \leqslant 1} |f^{(8)}(x)|,$$

$$f_1(x) = \frac{x^4(1-x)^4 + (2/5)\,x^3(1-x)^3 + x^2(1-x^2)/5 + x(1-x)/5}{8!}. \tag{1.18}$$

Note. If $d_p = \max_{0 \leqslant x \leqslant 1} |f_1^{(p)}(x)|$, then it can be verified that

$$d_0 = \left(\frac{93}{1280}\right)\frac{1}{8!}, \qquad d_1 = \left(\frac{1}{5}\right)\frac{1}{8!},$$

$d_{p+2} = \beta_p$, $p = 0, 1, ..., 5$, where the β_p are defined by (1.5). We denote by $k_3[f, x]$ the unique polynomial of degree $\leqslant 3$ satisfying

$$\begin{aligned} k_3[f, 0] &= f(0), \qquad & k_3[f, 1] &= f(1), \\ k_3[f, \tfrac{1}{2}] &= f(\tfrac{1}{2}), \qquad & k_3'[f, \tfrac{1}{2}] &= f'(\tfrac{1}{2}). \end{aligned} \tag{1.19}$$

We shall refer to $k_3[f, x]$ as quasi-Hermite interpolation polynomials. Concerning $k_3[f, x]$, we shall prove the following

THEOREM 4. *Let $f \in C^4[0, 1]$, let $k_3[f, x]$ be the unique polynomial of degree $\leqslant 3$ satisfying* (1.19). *Then we have, for $p = 0, 1, 2, 3$,*

$$|e^{(p)}(x)| = |f^{(p)}(x) - k_3^{(p)}[f, x]| \leqslant \nu_p \max_{0 \leqslant x \leqslant 1} |f^{(4)}(x)|, \tag{1.20}$$

where

$$\nu_0 = \frac{1}{1536}, \qquad \nu_1 = \frac{1}{96}, \qquad \nu_2 = \frac{5}{48}, \qquad \nu_3 = \frac{1}{2}. \tag{1.21}$$

Furthermore, these constants are best possible as can be verified by choosing

$$f(x) = \frac{x(1-x)(1-2x)^2}{96}.$$

It seems that the following conjecture concerning (1.1) may be worth mentioning. Let $f \in C^{2m}[0, h]$ and $v_{2m-1}(x)$ be the unique Hermite interpolation polynomial of degree $\leqslant 2m-1$ matching u and its first $m-1$ derivatives $u^{(j)}$ at 0 and h. Then

$$|u^{(p)}(x) - v_{2m-1}^{(p)}(x)| \leqslant uh^{2m-p} \max_{0 \leqslant x \leqslant h} |f_2^{(p)}(x)|, \qquad p = 0, 1, \ldots, (2m-1),$$

where

$$u = \max_{0 \leqslant x \leqslant h} |u^{2m}(x)|, \qquad f_2(x) = \frac{x^m(h-x)^m}{2m!}.$$

The above conjecture is true for $m = 2$, and $m = 3$. For other related interesting results, see [2].

2. PRELIMINARIES

Let us denote by $L_{2m-1}(u, x)$ the interpolation polynomial of degree $\leqslant 2m-1$ satisfying the conditions

$$L_{2m-1}^{(2p)}(u, 0) = u^{(2p)}(0), \qquad L_{2m-1}^{(2p)}(u, 1) = u^{(2p)}(1), \qquad p = 0, 1, \ldots, m-1. \tag{2.1}$$

The explicit formula for $L_{2m-1}(x)$ is given by

$$L_{2m-1}(u, x) = \sum_{i=0}^{m-1} [u^{(2i)}(1) \Delta_i(x) + u^{(2i)}(0) \Delta_i(1-x)], \tag{2.2}$$

where

$$\Delta_i(x) = \frac{2^{2i}}{(2i+1)!} B_{2i+1} \frac{(1+x)}{2}, \qquad \text{for} \quad i \geqslant 1. \tag{2.3}$$

Here $B_n(x)$ denote the well known Bernoulli polynomials:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k B_{n-k}, \tag{2.4}$$

$$B_l = \sum_{k=0}^{l} \binom{l}{k} B_k, \qquad B_0 = 1. \tag{2.5}$$

From the properties of Bernoullis polynomials it follows that

$$\Delta_i''(x) = \Delta_{i-1}(x), \quad \Delta_i(0) = 0, \quad \Delta_i(1) = 0, \quad i \geqslant 1, \quad \Delta_0(x) = x. \tag{2.6}$$

Since $L_{2m-1}(u, x) \equiv u(x)$ for $u(x) \in \pi_{2m-1}$ (π_{2m-1} denotes the class of polynomials of degree $\leqslant 2m-1$) it follows from the Peano theorem that for $u \in C^{2m}[0, 1]$

$$e(x) \equiv u(x) - L_{2m-1}(u, x) = \int_0^1 G_m(x, t)\, u^{(2m)}(t)\, dt, \tag{2.7}$$

where $G_m(x, t)$ is the Peano-kernel. Following Widder [5], we have

$$G_m(x, t) = \int_0^1 G_1(x, y)\, G_{m-1}(y, t)\, dy, \qquad m = 2, 3, \ldots, \tag{2.8}$$

where $G_1(x, t)$ is defined by (1.8).

3. Proof of Theorem 1

Following the notation used by Birkhoff and Priver [1], we shall denote

$$G_m^{(i,j)}(x, t) = \frac{\partial^{i+j} G_m(x, t)}{\partial x^i\, \partial t^j}.$$

Now on using (2.7) we have

$$e^{(2j)}(x) = u^{(2j)}(x) - L_{2m-1}^{(2j)}(u, x) = \int_0^1 G_m^{(2j,0)}(x, t)\, u^{2m}(t)\, dt. \tag{3.1}$$

Let us substitute $u(x) = Q_{2m}(x)$ (as defined by (1.6)) in (3.1) and use various properties of $Q_{2m}(x)$ as stated in (1.9), we then obtained

$$Q_{2m}^{(2j)}(x) = (-1)^j Q_{2m-2j}(x) = (-1)^m \int_0^1 G_m^{(2j,0)}(x, t)\, dt. \tag{3.2}$$

Also from (2.8) and (1.8) it follows that $(-1)^n G_n(x, t)$ is nonnegative in the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant 1$. Further, from (2.8) it also follows that

$$G_m^{(2j,0)}(x, t) = G_{m-1}(x, t); \qquad G_m^{(2j,0)}(x, t) = G_{m-j}(x, t). \tag{3.3}$$

Therefore $(-1)^{m-j} G_m^{(2j,0)}(x, t) = (-1)^{m-j} G_{m-j}(x, t) > 0$, in the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant 1$. Hence, on using (3.1), (3.3) it follows that

$$\begin{aligned} |e^{(2j)}(x)| \leqslant u \int_0^1 |G_m^{(2j,0)}(x, t)|\, dt = u \left| \int_0^1 G_m^{(2j,0)}(x, t)\, dt \right| \\ = uQ_{2m-2j}(x). \end{aligned}$$

This proves (1.11). Next, we turn to prove (1.12). Due to (3.3) it is enough to prove (1.12) for $j = 1$. From (2.8) it follows that

$$G_m^{(1,0)}(x, t) = \int_0^x yG_{m-1}(y, t)\, dy + \int_x^1 (y-1)\, G_{m-1}(y, t)\, dy.$$

Therefore

$$\begin{aligned} \int_0^1 |G_m^{(1,0)}(x, t)|\, dt \leqslant \int_0^1 \int_0^x y\, |G_{m-1}(y, t)|\, dy\, dt \\ + \int_0^1 \int_x^1 (1-y)\, |G_{m-1}(y, t)|\, dy\, dt. \end{aligned} \tag{3.4}$$

From (3.2) we know that

$$Q_{2m-2}(y) = \int_0^1 |G_{m-1}(y, t)|\, dt. \tag{3.5}$$

On changing the order of integration in (3.4) and making use of (3.5) we obtain

$$\begin{aligned} \int_0^1 |G_m^{(1,0)}(x, t)|\, dt \leqslant \int_0^x yQ_{2m-2}(y)\, dy + \int_x^1 (1-y)\, Q_{2m-2}(y)\, dy \\ \equiv \chi_{2m-2}(x). \end{aligned} \tag{3.6}$$

Now, we note that (on using (1.9))

$$\begin{aligned}\chi_{2m-2}(x) &= -\int_0^x yQ''_{2m}(y)\,dy - \int_x^1 (1-y)\,Q''_{2m}(y)\,dy \\ &= (-2x+1)\,Q'_{2m}(x) + 2Q_{2m}(x).\end{aligned} \tag{3.7}$$

Also

$$\chi'_{2m-2}(x) = (1-2x)\,Q''_{2m}(x) = (-1+2x)\,Q_{2m-2}(x).$$

Since $Q_{2m-2}(x)$ vanishes only at $x=0$ and $x=1$, it follows that the critical points of $\chi_{2m-2}(x)$ are $x=0$, $x=1$, $x=\frac{1}{2}$. Also $\chi_{2m-2}(1)=\chi_{2m-2}(0)$ and

$$\chi_{2m-2}(1) - \chi_{2m-2}(\tfrac{1}{2}) = \int_{1/2}^1 (2y-1)\,Q_{2m-2}(y)\,dy > 0.$$

Thus we conclude that $\chi_{2m-2}(x)$ has an absolute maximum at $x=0$ and $x=1$. Therefore, from (3.6) and (2.7) it follows that

$$\begin{aligned}|e'(x)| &\leqslant u\int_0^1 |G_m^{(1,0)}(x,t)|\,dt \leqslant u\chi_{2m-2}(x) \\ &= [(1-2x)\,Q'_{2m}(x) + 2Q_{2m}(x)]u \\ &\leqslant u\chi_{2m-2}(1).\end{aligned}$$

But from (3.7) it follows that

$$|e'(x)| \leqslant u\chi_{2m-2}(1) = -uQ'_{2m}(1) = uQ'_{2m}(0).$$

This proves (1.12) completely.

The inequalities (1.11) and (1.12) are both best possible. To show this, take $u(x)=Q_{2m}(x)$, the Euler polynomial defined by (1.6) and (1.7). In view of (1.9), we have $\max_{0\leqslant x\leqslant 1}|u^{2m}(x)|=1$. Further use of (1.9) and the definition of $L_{2m-1}(u(t),x)$ given by (1.10) shows at once that $L_{2m-1}[Q_{2m}(t),x]\equiv 0$. Now it is easy to verify that (1.11) is indeed best possible pointwise. A similar argument shows that (1.12) is also best possible. Here again we use the same choice of $u(x)$, namely, $Q_{2m}(x)$.

4. Proof of Theorem 2

Let $f\in C^6[0,1]$ and $H_5[f,x]$ be the unique polynomial of degree $\leqslant 5$ satisfying (1.13). We set

$$e(x) = f(x) - H_5[f,x], \tag{4.1}$$

and note that

$$e^{(p)}(0)=0, \qquad e^{(p)}(1)=0, \qquad p=0, 2, 3. \tag{4.2}$$

Thus $e(x)$ can be looked upon as the solution of the differential equation

$$\frac{d^6y}{dx^6}=Q(x)\equiv f^{(6)}(x) \tag{4.3}$$

subject to the boundary conditions

$$y^{(p)}(0)=y^{(p)}(1)=0, \qquad p=0, 2, 3. \tag{4.4}$$

We may express (4.3), (4.4) as

$$\begin{aligned} &\frac{d^2y}{dx^2}=\chi(x),\\ &y(0)=0, \qquad y(1)=0 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\frac{d^4\chi}{dx^4}=Q(x),\\ &\chi(0)=\chi(1)=\chi'(0)=\chi'(1)=0. \end{aligned} \tag{4.6}$$

From (4.5) it follows that

$$y(x)=\int_0^1 G_1(x, z)\,\chi(z)\,dz, \tag{4.7}$$

where the $G_1(x, z)$ is Green's function defined by (1.8). Also, the solution of (4.6) is known from the work of Birkhoff and Priver [1]. It is given by

$$\chi(x)=\int_0^1 G_4(x, t)\,Q(t)\,dt, \tag{4.8}$$

where

$$\begin{aligned} 6G_4(x, t) &= (3t^2-2t^3)\,x^3+3(t-2)\,t^2x^2+3t^2x-t^3, && t\leqslant x,\\ &= (3t^2-2t^3-1)\,x^3+3(1-t)^2\,tx^2, && t\geqslant x. \end{aligned} \tag{4.9}$$

Therefore,

$$y(x)=\int_0^1 G(x, t)\,Q(t)\,dt=\int_0^1 G(x, t)f^{(6)}(t)\,dt, \tag{4.10}$$

where

$$G(x, t) = \int_0^1 G_1(x, z)\, G_4(z, t)\, dz. \tag{4.11}$$

Since $G_1(x, z)$ and $G_4(z, t)$ do not change sign, it follows that $G(x, t)$ is non-negative in the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant 1$. Now, using a familiar argument, it follows that

$$G^{(2,0)}(x, t) = G_4(x, t) \tag{4.12}$$

and

$$G^{(l+2,0)}(x, t) = G^{(l,0)}(x, t), \qquad l = 0, 1, 2, 3. \tag{4.13}$$

From the known results of [1],

$$\max_{0 \leqslant x \leqslant 1} \int_0^1 |G_4^{(l,0)}(x, t)|\, dt = \alpha_l, \qquad l = 0, 1, 2, 3. \tag{4.14}$$

From (4.13) and (4.14), (1.14) follows for $p = 2, 3, 4, 5$. Thus it remains to prove (1.14) for $p = 1$. For this purpose we need to compute $\max_{0 \leqslant x \leqslant 1} \int_0^1 |G^{(1,0)}(x, t)|\, dt$. On using (4.11) we obtain

$$G^{(1,0)}(x, t) = \int_0^x y G_4(y, t)\, dy + \int_x^1 (y - 1)\, G_4(y, t)\, dy.$$

Therefore

$$|G^{(1,0)}(x, t)| \leqslant \int_0^x y\, |G_4(y, t)|\, dy + \int_x^1 (1 - y)\, |G_4(y, t)|\, dy$$

and we know

$$\int_0^1 |G_4(y, t)|\, dt = \frac{y^2(1 - y)^2}{4!}.$$

Thus we can write

$$\int_0^1 |G^{(1,0)}(x, t)|\, dt \leqslant \int_0^x \frac{y^3(1 - y)^2}{4!}\, dy + \int_x^1 \frac{(1 - y)^3\, y^2}{4!}\, dy = \theta(x).$$

But

$$\theta'(x) = \frac{x^2(1 - x)^2(2x - 1)}{4!}.$$

Thus $\theta(x)$ has only three critical points: $x=0$, $x=1$, $x=1/2$. Since $\theta(1)>\theta(1/2)$ it follows that

$$\max_{0\leqslant x\leqslant 1}\int_0^1 |G^{(1,0)}(x,t)|\,dt \leqslant \int_0^1 \frac{y^3(1-y)^2}{4!}\,dy = \frac{1}{1440}.$$

This proves (1.14) for $p=1$ as well. Proof of Theorem 3 is very similar to the proof of Theorem 2 we will not give any details. Proof of Theorem 4 can be given on the lines of Theorem A so we will not give the details.

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